Discourse in mathematics pedagogical content knowledge

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What is happening for in-service teachers at the classroom intersection of mathematics, culture(s), teaching, and learning? How can knowing the answer to that question inform teacher preparation, induction, and development? In ongoing efforts to model and measure the intercultural and relational aspects of pedagogical content knowledge, we present a model and data analyses. The focus is teacher learning and intercultural orientation development. Data are pre- and post-program written tests, surveys, and classroom observations among four cohorts (70 in-service teachers) enrolled in a two-year master's program. The focus at the conference was harvesting the intellectual power of the audience to consider questions about the connections – qualitative, quantitative, and otherwise – among core constructs in pedagogical content knowledge, the thinking that teachers do in connecting them, and how knowing about intercultural orientation and how it plays out in the classroom can inform teacher education and professional development.

Key words: Pedagogical content knowledge, Discourse, Intercultural awareness

Background

What mathematical reasoning, insight, understanding, and skills are entailed when a person teaches mathematics well? Many have worked to develop theoretical models and measures to address this question (Ball, Thames, & Phelps, 2008; Hill, Ball, & Schilling, 2008; Shulman, 1986). In their work, Ball and colleagues have proposed three types of subject matter knowledge and three types of pedagogical content knowledge (PCK) as non-overlapping categories in the domain of mathematical knowledge for teaching (MKT, see Figure 1, next page).

Current U.S. educational policy requires evidence-based decisions about teacher preparation, induction, and development. Meeting this need calls for models and measures that are credible and transferable across at least a small range of mathematics instructional contexts. The MKT model and related instrument development for K-8 teachers have provided a reliable and useful foundation at these lower grades. Ongoing development of MKT models for grades 8 and higher is adding to that foundation (Hauk, Toney, Jackson, Nair, & Tsay, 2014; Speer, King, & Howell, 2014). These additions at the secondary and post-secondary level have focused on mathematical discourse and meaning-making for teaching (Hauk et al., 2014; Powers, Hauk, & Goss, 2013; Speer et al., 2014; Thompson & Carlson, 2013). Thought, speech, and context inform each other. In particular, struggling with the ambiguities introduced in learning to use technical vocabulary, in and out of classroom contexts, supports mathematical meaning-making

(Barwell, 2005). In parallel, developments in teacher education research have included calls for attention to the cultural and sociopolitical aspects of mathematics instruction (e.g., Gutiérrez, 2012, 2013). The knowing that happens in pedagogical content knowledge can be seen as both a set of connections among rather stable fact-sets and as contextualized, but dynamic, ways of thinking.

Discourse, as an aspect of teaching, is central in our effort to bring an explicit attention to the use of language and the dense set of values about mathematical appropriateness, clarity, and precision that are integral to thinking, learning, and communicating in mathematics both in and out of school settings. Our previous work has discussed the connection between Ball and colleagues' model of PCK and an additional aspect called *knowledge of discourse* that relies on ideas from intercultural orientation (Hauk, et al., 2014). Here we report on our continuing work to address the twin needs of measures that capture information about PCK and models that attend to the actively cross-cultural nature of most mathematics instruction in the U.S. Hinging on unpacking "discourse" and connecting it to the PCK model shown in Figure 1, this work has led to the model in Figure 2.

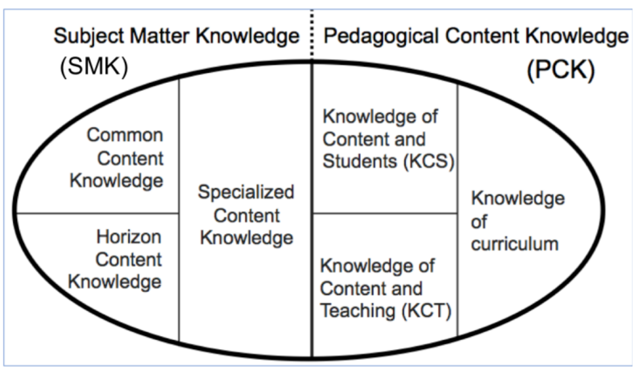

Figure 1. Dimensions of mathematical knowledge for teaching (MKT) from Hill, et al. (2008).

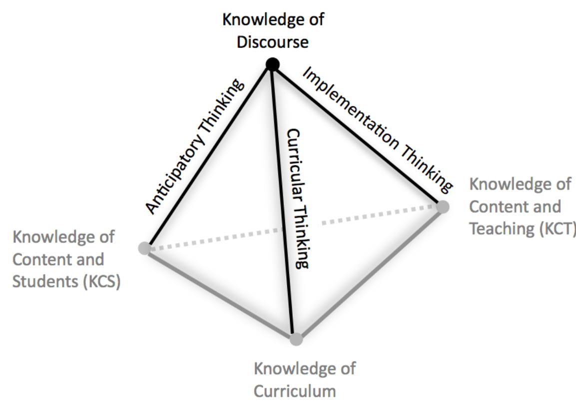

Figure 2. Extended model of PCK, from Hauk, et al. (2014).

The development of the model in Figure 2 has been grounded in classroom practice. The need for a construct like Knowledge of Discourse emerged early in our efforts to develop a measure of PCK that would capture growth in the kinds of knowledge valued as a mathematics teacher builds instructional effectiveness. Across our work, secondary and post-secondary teachers have said they know they are effective when students learn facts and, also, build a flexible understanding of mathematical ideas that can be brought to mind and actively used when needed. Early assessment and interview development led us to reuse that as a description of how to know that professional development was effective: We know professional development is effective when teachers learn facts and, also, build a flexible understanding of MKT ideas that can be brought to mind and actively used when needed. It was in getting at the "brought to mind and actively used" aspect that Knowledge of Discourse came to the foreground.

Throughout the revisions of the model summarized so briefly in Figure 2, we have iteratively visited three major strands of work:

Area 1. developing a written test that can capture change in PCK,

Area 2. advancing work on an observation-plus-interview protocol that can document bringing to mind and using PCK, in real time in the classroom, and

Area 3. refining a model of PCK to provide language, and examples, for our own further development as teacher educators and researchers in mathematics education. Each of these aspects has contributed to this report. Below, after providing some background on the model, we offer information on empirical results in a particular project related to Areas 1 and 2. These were shared at the conference presentation as background for a lively conversation about Area 3. We close with the fruits of the RUME conference discussions and some thoughts on next steps.

Theoretical Framework

In his review of over 100 research publications in mathematics education that reported on "discourse," Ryve (2011) found that the myriad conceptions of discourse offered by researchers could be understood through the work of Gee (1996), who distinguished between "little d" discourse and "big D" Discourse. "Little d" discourse is about language-in-use. In mathematics teaching and learning, this may include connected stretches of utterances and other agreed-upon ways of communicating mathematics such as symbolic statements or diagrams. Discourse (big D) is situated discourse, encompassing verbal and non-verbal aspects, from the subtleties of local vocabulary and symbolic or diagrammatic representation to the nuances of gesture, tone, hesitation, wait time, facial expression, hygiene, and other aspects that make for authenticity in an interaction (Gee, 1996). In what follows, our use of the term *discourse* is in the "big D" sense. Discourse, so defined, addresses Shulman’s (1986) attention to semiotics:

> The syntactic structure of a discipline is the set of ways in which truth or falsehood, validity or invalidity, are established... Teachers must not only be capable of defining for students the accepted truths in a domain. They must also be able to explain why a particular proposition is deemed warranted, why it is worth knowing, and how it relates to other propositions, both within the discipline and without, both in theory and in practice… This will be important in subsequent pedagogical judgments. (p. 9)

As indicated in the excerpt above, Shulman’s original statements about pedagogical content knowledge included knowledge for interacting effectively with the multiplicity of discourses students, teacher, curriculum, and school bring into the classroom. In particular, in the cultures of secondary and post-secondary academic and research mathematics, valued communication includes (among others) the sense-making discourse practices of description, explanation, and justification. These are also valued in school mathematics curriculum and instruction (e.g., the *Common Core* Standards for Mathematical Practice, National Governors Association, 2010).

The ways that teachers and learners are aware of and respond to valued forms of communication across multiple cultures is a consequence of their orientation towards cultural difference, their *intercultural orientation*. This is not a reference to teacher beliefs about the teaching and learning of mathematics. Rather, intercultural orientation is the perspectives about *difference* each person brings to interacting with other people, in context. For teachers, it includes perceptions about the differences between their own views and values around teaching and learning and the views of their students.

Gutiérrez (2013) refers to *conocimiento* to identify a relational, connected, way of knowing that is qualitatively different from declarative kinds of knowing (e.g., of facts and their contexts). Our work, too, relies on this idea and it is reflected in the "thinking" edges of the PCK model in Figure 2. What is more, Gutiérrez’s (2012) *Nepantla* captures the aspects of professional learning Shulman described as "the exercise of judgment under conditions of unavoidable uncertainty" and the "need for learning from experience as theory and practice interact" (Shulman, 1998, p. 516), both of which are aspects of the interculturally informed discourse extension to the model of PCK. We join an already moving river of ideas. Various streams of research and development on mathematics teacher learning already spring from a research-practice synergy that views all people in a classroom as participants in learning. It is the question of the nature of that learning and of the interaction of the people in its support that is foundational (Schoenfeld, 2013).

Though some teachers work in largely monocultural classrooms – in the sense that most students share experience of a common set of culture-general norms and practices – the U.S. is shifting from such circumstances to cultural heterogeneity. For example, the 21st century version of multicultural can mean 2, 5, or even 10 different home language groups in a single classroom (Aud, Fox, & KewalRamani, 2010). Given the diversity of students in the nation's classrooms and the demographics of instructional staff in U.S. schools, teachers are destined to have regular opportunities for cross-cultural classroom experience that, for most, will be fraught with unavoidable uncertainty. Many new teachers leave, citing as a reason that they were not prepared for what the work is really like (Keigher, 2010).

What was recently explored in the project from which this research emerges is attention to this missing aspect of heterogeneity: dealing with the realities of navigating the multiple cross-cultural relationships in professional development and school contexts. Several frameworks exist for interacting and communicating with people across professional (and personal) cultures. In particular, healthcare and international relations have generated suggestions based on theories of intercultural sensitivity development and styles of conflict resolution communication (e.g., Bennett, 2004; Hammer, 2009). The developmental model of intercultural sensitivity centers on orientations towards cultural difference (Bennett, 2004). The core of this approach is building skill at establishing and maintaining relationships in, and exercising judgment relative to, interculturally-rich situations. The developmental continuum has five named milestone orientations to noticing and making sense of cultural difference: *denial*, *polarization*, *minimization*, *acceptance*, and *adaptation*. With mindful experience we develop from ethno-centric ignoring or *denial* of differences, moving through an equally ethno-centric *polarization* orientation that views the world through an us-versus-them mindset. With growing awareness of commonality, we enter the less ethno-centric orientation of *minimization*, which may, however, over-generalize sameness and commonalities. From there, development leads to an ethno-relative *acceptance* of the existence of intra- and intercultural differences, and on to a highly ethno-relative *adaptation* orientation.

Discourse is situated, in the present case it is situated in a mathematics class, and Knowledge of Discourse includes what a teacher may say. It also is used in how the teacher orchestrates conversation and discussion in the classroom. And, it is about what a teacher knows or anticipates about students' previous experiences and how to situate that in the classroom -- in the context of the mathematics goals in the classroom. For example, knowing how to establish, elicit, and respond to sociomathematical norms, would live in Knowledge of Discourse.

The lens of intercultural orientation development leverages powerful agents for improving teaching and collegial interaction. Teachers can build self-awareness and apply developmentally (for them) appropriate approaches to their own learning with colleagues and to student learning in their classrooms. We return to our exploration and development of examples of these ideas (Area 3 of our research program) after first sharing a brief summary of some empirical results related to Area 1 and Area 2. The empirical work is to provide some of the context in which the model in Figure 2 was developed (and continues to be revised) and a foundation for the reporting on the results of the intellectual work of the group at the RUME conference session.

Written Test and Observation/Interview Protocols - Measuring PCK

Methods

Setting: The setting was a blended face-to-face and online delivered master's degree program in mathematics for in-service secondary teachers. Designed to reach urban, suburban, and isolated teachers in rural areas, the program is conducted using a variety of technologies (e.g., *Collaborate* for synchronous meetings, *Edmodo* for asynchronous communication). Offered through a joint effort at two Rocky Mountain region universities, cohorts of 10 to 20 participants complete a 2-year master's program in mathematics with an emphasis in teaching (about half of the course credits in mathematics, half in mathematics education).

Participants: Participants for the quantitative results reported here were in-service secondary teachers who teach grades 6 to 12 mathematics. To date 71 teachers have entered the program, 33 have completed it, 18 are continuing, and 20 have dropped or taken leave from the program.

Instruments: The development of the written test of pedagogical content knowledge and real-time observation instrument is reported elsewhere (Hauk, Jackson, & Noblet, 2010; Jackson, Rice, & Noblet, 2011). The most important things to note here are that the written assessment included: released items from the LMT (Ball et al., 2008), new items with more complex mathematical ideas modeled on the LMT items, some secondary *Praxis* items, and open-ended extensions to these limited option items. Multi-year test development has included cognitive interviews with in-service teachers and mathematics teacher educators as they completed individual items or collections of items. In addition to the established face validity of the tests, tests of the constructs' internal consistency (Cronbach's alpha) indicate good overall reliability (α >.75 on each construct).

Constructs on the written instrument were curricular thinking, anticipatory thinking, and kinds of Knowledge of Discourse. While the written test of PCK has included items related to KCT, as a component of implementation thinking, testing this knowledge by self-report is problematic. So far, it has seemed that a better way to get rich information about implementation thinking is through observing a teacher in the classroom and interviewing about the observation later. The observation instrument documented in-class actions, utterances, and behaviors related to curricular thinking, anticipatory thinking, implementation thinking, and kinds of Knowledge of Discourse (e.g., observation categories included noting instances of mathematical description, mathematical explanation, mathematical justification – more on this below).

As of this writing, we have pre-tests for 70 teachers, first follow-up tests for 61 teachers (after 1 year in the program), and exit exams (post-program) for 33 teachers. Also at this writing, pre- and post-program observation data is complete for 17 teachers. The observation instrument, based on the LMT video observation protocol (see Learning Mathematics for Teaching website; development reported elsewhere) showed good reliability overall (α >.78 on each construct). Like the LMT protocol, the observation tool used samples called "segments" (6 minutes each: 3 minutes observed, 3 minutes to record notes; each class visit had 7 to 12 segments). An "observation" was three consecutive classroom visits. Experienced observers trained new observers to use the instrument; inter-rater reliabilities were greater than 0.8. To measure intercultural orientation and sensitivity development we used the established Intercultural Development Inventory (Hammer, 2009; idiinventory.com).

Empirical Results

We care about generating research-based and theory-grounded quantitative results because school leaders have to make evidence-based decisions about teacher learning. Current policy says "evidence" is based on test results. Reciprocally, what the empirical study is giving us is nuanced examination of teacher knowledge growth, in service of theory and model development.

Results after four years have indicated teacher knowledge growth for each of the constructs of interest. Paired samples *t*-tests on teachers' percent scores on the written test indicate statistically and practically meaningful growth in the desired direction in curricular content knowledge and discourse knowledge. Teachers' scores on items coded as Knowledge of Discourse (KofD) increased significantly (t=2.189, p=.047) from pre-test (M=56.82, SD=15.43) to post-test (M=66.22, SD=19.09).

For the observation data, to date there are two statistically significant results (Bonferroni correction applied). One was in the observation category "General language for expressing mathematical ideas (overall care and precision with language)." While such use of general language was seen, on average, in about 49% of pre-program classroom segments, by the end of the program it was present in more than 80% (M=80.34, SD=19.71). The other significant result was in "Mathematical descriptions (of steps)" (i.e., segments where the teacher or students accurately used mathematical language – in symbols, words, shapes, or diagrams – to describe the steps of some process). On average, across pre-program observations, this was seen in about 40% of class segments (M=40.28, SD=21.94), increasing to almost 70% of segments, post-program (M=68.10, SD=19.31). Though not statistically significant, there was also increase in the relative frequency of mathematical explanations (from 40% to 51%) and justifications (14% to 23%). Three other observed variables appeared to be approaching significance (i.e., p<.01): the percent of segments where (a) student voices were present in the room (increasing from 80% to 90% of segments), (b) teachers were observed to use conventional notation (increasing from 54% to 90% of segments), and (c) fewer mathematical errors occurred (decreasing from about 4% of the time to nearly 0%). Similarly, we have seen changes in the desired direction on the measure of intercultural competence development (e.g., see Hauk, Yestness, & Novak, 2011).

Examples - Communicating in and through PCK

The ways teachers responded to PCK test items and their extensions (on paper and in cognitive interviews) led to questions for us related to discourse (little d) and, eventually, to big D discourse. To illustrate, we give two examples. First, we present an example that highlights the connection between intercultural orientation and Knowledge of Discourse. Then, a second example takes the form of an annotated script, a fictionalized version, based on an actual conversation between two teachers (one a novice and one more experienced) as they worked through a task from the written test of PCK.

Example 1: Coexistence of Mathematics and Physics Discourses in Calculus

In our current work to unpack Knowledge of Discourse we consider the continuum of intercultural orientation, of ways of seeing differences between one's own values, view, and communication of the (mathematical) world and that of others. Central to this idea of intercultural awareness is ways of noticing. Perhaps the *denial* orientation might take the form (in the context of mathematics instruction): "I know the MATH, the math discourse, I don't really notice any other discourse." Such an orientation is not of denial in the sense of "I'm going to say it is not there" but denial as in "I can't even see it."

The *polarization* orientation towards orchestrating the conversation in a math class might be characterized as: "There's a RIGHT way to talk about things and there's a WRONG way to talk about things. And we're going to make sure we use the right way." Depending on the experience and values of a teacher, the "right" way to talk about applied related rates problems in calculus may or may not include physics discourse or associated engineering discourse. Nonetheless, enacting a polarized orientation in mathematics teaching would mean seeing, for instance, that a mathematical practice is happening or noticing a norm being developed. Perhaps, when a teacher strongly identifies with the mathematical culture, they are loyal to that culture. And, when focused on right ways and wrong ways of talking, do not attend to (may not really care) what is done in a physics class.

From a *minimization* orientation, minimizing differences and paying attention to similarities, teachers may also be very true to their mathematics knowledge, their mathematical culture, and valued ways of communicating. Yet, for someone mathematically trained, this might be characterized as, "Look how this is LIKE mathematics. Physics is like mathematics, the idea is similar even if the way it is said is a little different. Let's talk about how it is similar. Let's leverage the fact that students have seen this in physics before." Consider a basic example in the representation of vectors. Suppose the book represents vectors in the form $v=3i+5j$ and some students, who are also in physics, write $v=<3,5>$. It may be characteristic of a minimization orientation to write both representations on the board *once* and then note "But these are basically the same, so we'll use the one I know, the one common in math." In development towards an *acceptance* orientation, it might be more characteristic to notice and accept either representation on students' written work and suggest students use whichever makes most sense for them – anchored in the idea of a common goal, that vectors make sense to students. Pushing this small example even further, a well-developed acceptance orientation might be evidenced when a teacher alternated between the notations when talking with students and encouraged students to become fluent in both (i.e., modeling fluency in moving back and forth among the different representations while also encouraging students to accept and understand the difference in the representations).

More generally, an *acceptance* orientation might be characterized by a statements like: "I'm a mathematician, but I'm accepting the fact that all of my students are not going to be mathematicians" and "I'm accepting the fact that there may be other ways, physics ways or biology ways, of talking about this mathematical idea that are valuable, and maybe even more valuable to them [the students] than my math way of talking about it. I'm going to embrace that, those various ways, coming out in the conversation in the classroom." But a general intention of accepting the different ways in the classroom may not provide guidance to students about how to make decisions on which discourse(s) are useful in a given mathematical context (e.g., solving applied problems in biology may not be facilitated by an abstract mathematics vocabulary, and vice versa).

A further developmental orientation is *adaptation*. Now, not only does a person accept that there are these differences, the adaptation oriented teacher seeks out ways to give students opportunities in noticing, articulating, and responding to those differences. An adaptation orientation might be characterized by statements such as: "I seek out ways to have students pursue opportunities that arise from variety in approach or strategy. I don't have to give many, or even one method to them. They can go get it. I don't have to be in the loop. So math is a relative thing now. Learning math is still central but, while the goals are for learning about rigorous math and include the standard math language and representations, how I and students connect ideas and access, or organize, or value ideas is not necessarily strictly limited to the ways valued by a purely mathematical perspective."

Though not fully delineated by researchers, the theory of intercultural competence development also hypothesizes something called an *integration* orientation. This is something that is likely to be very rare. This perspective might be characterized by a statement like: "Okay, that physics approach to this problem is a whole other way of looking at the world. It's internally consistent. Which I, as a mathematician, value. So, it's okay. And I'm going to integrate what I can without violating my own truth to mathematics. I'm going to be myself as a mathematician, in that environment." We suspect such a view might be analogous to the ultimate mission behind much of theology: studying a variety of belief systems, without disagreement or approval of the system, while remaining authentic in one's own beliefs. In the research around intercultural competence development, examples of how an integration orientation might be realized come in the shape of expert and effective negotiators in high stakes endeavors (e.g., diplomat, hostage negotiator).

Example 2: Discourse During Use of Pedagogical Content Knowledge

Bringing to mind and using mathematical knowledge for teaching happens in many ways. An example of *curricular thinking* in the model in Figure 2 comes when mathematically situated discourse and knowledge of curriculum are brought to mind to create a rubric for grading a quiz. Among the items appearing on the PCK written test, was a task that asked teachers to do a mathematics item and then to generate a rubric for grading the item. The conversations that follow were based on actual teacher work and cognitive interview. First we generated a 2-column conversation of "little d" discourse – the actions and utterances of two teachers, Selma (experienced) and Jamie (novice) in solving the problem (this material can be seen in the table of the interaction below in column 1 and the bold face material in columns 2 and 3). Then, based on cognitively guided interviews on the task, we created the extensively annotated 3-column example, sketching the thoughts of each teacher. The first part of the interaction is focused on subject matter knowledge, SCK in particular. The balance is about their work to make a rubric. The purpose here is to formalize an example. It is based on the needs that emerged from conference attendees' wrestling with the ideas presented. The example is meant as an illustration of why it matters and can be useful to consider various aspects of Knowledge of Discourse in teacher education, induction, and professional development.

> Part 1: The Richter scale is a base 10 logarithmic scale used to measure the magnitude of earthquakes; i.e., an earthquake measuring 7 is ten times as strong as an earthquake measuring 6. An earthquake that measures 6.8 on the Richter scale has a magnitude that is approximately what percent of an earthquake measuring 6.6?
>
> Part 2: Provide a rubric that you could use for grading student answers.

Figure 3. Test item with extension.

Selma's ethno-centric approach to noticing and dealing with difference, a polarization orientation to difference, is represented in her view that her own knowledge of mathematics is paramount in solving the problem, and that she must compare whatever Jamie says to that foundation. For each of Jamie's contributions, Selma must determine whether Jamie is with her (therefore right, part of "us") or not (therefore wrong, part of a different group or "them"). Elements of this are evidenced in her "I" language in rows 5, 16, and 18, and in Selma regularly pausing the problem solving process to evaluate whether suggestions are right or wrong (rows 9, 12, 16, 18). Jamie, whose orientation is to minimize difference, views her knowledge as being essentially the same as Selma's. For Jamie, because they both "speak mathematics," it will not be difficult to work together to solve the problem. She interprets Selma's comment in row 5 as affirming "their" problem solving process, and shifts to "we" language (rows 10, 13, 17).

The interaction also has evidence of orientation in the approach each takes to (a) creating and (b) defending decisions about generating a rubric. Still focused on using her knowledge as the central reference, Selma asserts that how she awards points in her rubric is different from Jamie's method. Meanwhile, Jamie works to find commonality between the two (row 29). Jamie maintains that they have an important commonality, the language of mathematics, though the specific wording may be different.

Early in the conversation, Jamie decides she and Selma are "on the same page" (row 2). She spends the next few lines confirming they are thinking the same way about the problem, even while Selma considers whether they might be thinking differently (rows 6 and 7). In fact, Jamie spends much of the conversation looking for ways to affirm her convictions that she and Selma are thinking similarly about the problem-solving context (rows 5, 20, 21) and in creating a rubric (rows 28, 29, 30, 31, 39). Selma, on the other hand, looks to see *if* she and Jamie are like-minded. Jamie confirms for her they are like-minded in the problem solving context (rows 14, 18). Once they begin the rubric task, Selma must again decide whether she and Jamie are like-minded. Given their initial rubrics (see Figures 4 and 5), she quickly decides they are not (rows 27, 28). Pointing out those differences gives rise to some tension. When encountering conflict, as when the social or emotional stakes go up, people tend to fall back to an earlier developmental orientation. This is represented in the vignette when Selma and Jamie revert to denial and polarization, respectively (rows 31-37).

	Description of actions while working on prompt	*Selma*	*Jamie*
1	The prompt is written on the center of the whiteboard. Both stand at the board, the prompt visible between them, calculators in hands.		**I'm first thinking of using logs because it says "base 10 log scale." But then I'm thinking we want to make a ratio because it says "10 times as strong."**
2	Selma picks up a marker and writes the following on the board: 10^6 $10\times$ 10^7	*"10 times as strong": If that's the information in the prompt, then we also need information about* $10^{6.8}$ *and* $10^{6.6}$.	*She's writing the ratio. We're thinking about the problem the same way. We're on the same page, so we'll proceed together. I don't have to think about that part anymore.*
3	Selma punches on the keypad of her calculator. She writes the following on the board under her previous figure: $10^{6.8} = 6309573.445$ $10^{6.6} = 3981071.708$	**So, we need…**	*[continuing to make sense of the prompt] If I have to figure out a way to solve this problem, percent is also going to be important.*
4	Jamie points at the prompt.	*To find the percent change, I do this procedure.*	**It says "percent." So, greater than 100%.**
5	Selma gestures at $10^{6.8} = 6309573.445$ $10^{6.6} = 3981071.708$	**If I subtract these two. Oh wait.**	*We have a shared knowledge of how to compute percents. I'm continuing with your procedure.*
6	Jamie enters "$10^{6.8} - 10^{6.6} =$" into her calculator. Then she enters "$Ans \div 10^{6.6} =$"	*Something about the prompt saying log scale makes me uncomfortable. I'm worried your way is not the right way.*	**And divide it by the 6 one.**
7	Jamie writes .58489 next to Selma's calculations of $10^{6.8}$ and $10^{6.6}$.	*I'm not sure that's right, but I'm going to see what you do. Maybe you are doing it the right way.*	**So, ".58489." 58%.**
8	Jamie points to 10^6 $10\times$ 10^7	*Something about the nature of percents is giving me pause. Are we computing these correctly?*	**6 is 10% of the 7 one, right?**
9	Selma steps back from the board.	**Is it?**	*What is 10% of* 10^7*?*
10	Jamie points to 10^7.	*Okay, I'm listening to you. That's the right*	**Well, if we times this one by .1.**

		way to compute the percent.	
11	Jamie looks down at her calculator and enters $10^{6.8} \times .58493 =$	*Something about the nature of percents is still making me uncomfortable. I'm not sure this problem is right. Does it want percent increase? Or percent change? What is the right answer?*	**So, $10^{6.8}$ times .58493 is 3,981,071. Okay, so $10^{6.6}$.**
12	Selma points to the prompt.	**Is this worded correctly? It has to be over 100. So, that's the percent increase. Would it be 158%?**	*We subtracted to find what percent more* $10^{6.8}$ *is than* $10^{6.6}$*. But the question asks what percent is 6.8 of 6.6?*
13	Jamie enters $10^{6.8} \div 10^{6.6} =$ into her calculator.	*Okay, percent change is the right question.*	**Were we supposed to subtract? We found the difference. So maybe it's just $10^{6.8} \div 10^{6.6}$. So, it's 158.5, which makes sense.**
14	Selma points to 10^6 $10\times$ 10^7	**Yeah. Like that.**	10^7 *is 10% more than* 10^6 *because* $10^7 \div 10^6 = .1$*. So,* $10^{6.8} \div 10^{6.6} = 1.58$ *says* $10^{6.8}$ *is 158% of* $10^{6.6}$*. Okay, so it's essentially the same either way.*
15		*I'm still not sure this is the right way to do this, though. The prompt says log scale. I'm worried your way is not the right way.*	**Which makes sense.**
16		**Because it's a log scale, I feel like it's a log somewhere. So, I don't think we're right.**	*That's the same thing, isn't it?*
17	Jamie gestures at $10^{6.8} = 6309573.445$ $10^{6.6} = 3981071.708$ *[first points to right side of equal and then to two exponents]*	*Okay, I agree with you about the log thing. Is percent change really the question?*	**But it's log base 10 that converts it to magnitude. So, if we were to take the log of the magnitude, it would give us the Richter scale.**
18		**Yeah, that makes sense. I keep going back to percent change not increase.**	*Because they mean essentially the same thing. It's just how the question is worded.*
19		*You did it the right way.*	**It was a 58% increase, which means 158%.**
20	Selma writes 158.5% under Jamie's	**Okay. That makes sense.**	*We solved the problem together. Yay us!*
	.58489.		
21		**So, now it says to write a rubric.**	*We just solved the problem together. We're going to write a rubric together, too.*
22	Jamie points at Part 2 of the prompt and reads aloud.	*It will take way too long to try to do this together. We need to do it separately first.*	**"That you and your colleague could use for grading student answers." Does that mean we should make it together?**
23		**Well, yes. But let's start separate. You make yours and I'll make mine, and then we'll come together**	*Okay. That way we can use any small differences in our rubric to make the final one stronger.*
24	Both are quiet for several minutes as they write on separate sides of the white board. Selma writes [Figure X below] Jamie writes [Figure X below]	*Getting the answer wrong doesn't get you any points. Setting the problem up wrong doesn't get you any points. If you set up the first part of the problem correctly, you can get 1 point. If you set that part up correctly, and recognize the correct ratio between* $10^{6.8}$ *and* $10^{6.6}$*, you get 2 points. And of course, you get full credit when you do all of it right.*	*Right or wrong, I want them to be able to explain why they did what they did. If they can get the right answer and explain why it's correct, that should get full credit. If they can't do any of that, they should get 0 points. But they might be able to explain the whole problem right, but then have something fall apart in the math at the end. That should get a lot of credit because that's better than just guessing the right answer, but not really being able to say why. So, that should get 1 point and the other should get 2 points.*

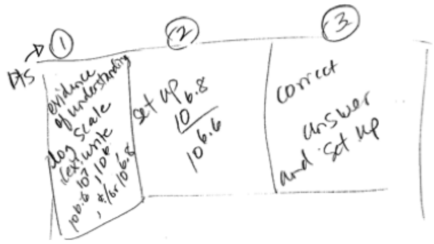

Figure 4. Selma's rubric.

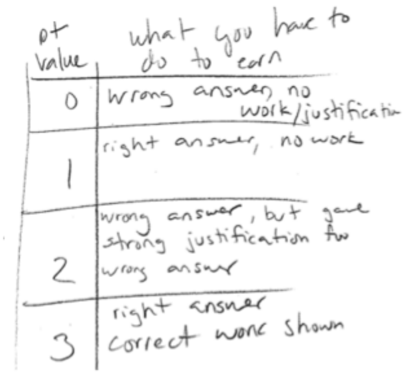

Figure 5. Jamie's rubric.

	Description of actions while working on prompt	*Selma*	*Jamie*
25	Selma steps back from the white board.	**Are you ready to talk?**	*We had the same idea about the math. We're probably thinking similarly about how to grade it.*
26	Jamie steps back from the white board and looks over at Selma's work.	*Okay, let's see what we did differently.*	**I think so.**
27	Selma looks over at Jamie's rubric.	*I'm already seeing big differences in these rubrics. She gives 2 points for a wrong answer and 1 point for a right answer. How can she give 2 points for wrong work?*	**Okay, this is what I did. I knew I wanted them to get the right answer.**
28	Jamie looks again at Selma's work and points to her 3-point column.	*Yes, those cells are the same, but there's still a lot of difference there.*	**Okay, like yours – and we both also want them to be able to explain it. Yeah, like you have "set up" and I have "correct work." So, a right answer with correct work or set up gets full credit.**
29	Selma points to the 1- and 2-point columns of her rubric.	**Right, but what I think is different is where we give 1 and 2 points. I'm basing all my points on how much of the problem they get right.**	*Really? You're showing what the "set-up" is on the rubric. That's essentially what I meant when I wrote "justification."*
30	Selma points to the 1- and 2-point rows on Jamie's rubric.	**You're giving points for a wrong answer. Why would you do that?**	*I don't think the wrong answer is what's important there. The justification is what's important. Like, if they wrote $\frac{10^{6.8}}{10^{6.6}}$ on their paper and then got the wrong answer for some reason. That's like what you wrote on your rubric.*
31		*But you still gave 2 points for the wrong answer!*	**We want to know what they *can* do. I know sometimes I start right, but then maybe I make a small mistake. But I knew what I was doing. I want to be able to give points to a student who made a small mistake.**
32		**Yeah. But why would a right answer get fewer points than a wrong answer?**	*I'm just saying a correct set-up might also sometimes get a wrong answer – like when they accidentally punch in the wrong numbers on their calculator.*
33	Jamie stands away from the white board and gestures back and forth between the two rubrics as she talks.	*Yeah. But my 1-point column doesn't have a place in your rubric. Just noticing powers of 10 isn't going to get 2 points when there's a lot more to the problem than that.*	**If I know they know how to set it up, I know they've got the foundations of the math we're teaching. That's what we're getting at in both of these rubrics. We're both looking at how they set up the problem. I'm just saying a correct set-up might also sometimes get a wrong answer – like when they accidentally punch in the wrong numbers on their calculator.**

34	Selma gestures at her rubric.	**Okay. But mine breaks down the steps of this problem into smaller chunks. With mine, the score is based on how much they know how to do. They get credit for doing each piece along the way.**	*What if they just write the right answer with no work?*
35		*That's not going to happen.*	**What if they just write the right answer with no work?**
36		**My students know better than to turn in a quiz without showing their work. If they got the right answer, the work will be right.**	*Eh, that's not an answer. We're going to have to agree to disagree here.*
37		*I don't think we're getting anywhere with this.*	*I don't think we're getting anywhere with this.*
38		*We need different rubrics because we're different teachers and we notice different things about this problem.*	**Maybe the rubrics don't have to be the same exactly if they're still getting at the same kinds of ideas.**
39		**Well, even when we give common tests and quizzes, we still grade our own stuff. I think we should have different rubrics.**	*What we have in common is that we agree that as different teachers we need different rubrics.*

Applications and Discussion

By adding Knowledge of Discourse as a variable to be described/measured, we include the interdependence of Knowledge of Discourse with KCS, KCT, and Knowledge of Curriculum in the extended model of PCK. The linking of these kinds of knowledge are represented through the connectors Anticipatory Thinking, Implementation Thinking, and Curricular Thinking, respectively, in Figure 2 (see Hauk et al., 2014 for more on these aspects).

Inevitably, there are both similarities and differences between teachers' own content-based acculturations, their own everyday cultures, prior mathematical enculturation of students, everyday culture of students, intended mathematical enculturation of the curriculum or school, and interim classroom cultures that combine all of these (and others, e.g., physics). The teacher having knowledge of these is mathematically important. Each has a mathematical component in terms of how one [student or teacher] sees mathematics or uses mathematics or values mathematics or communicates mathematically. And at the same time, for other disciplines it also is important. A rich Knowledge of Discourse in the context of calculus can include a knowledge of physics discourse (see, for example, the report in these proceedings by Firouzian & Speer, 2015). In fact, emergent from the conference presentation were conversations about the ways some knowledge of how those steeped in physics talk about and make sense of applied calculus problems is needed in order for a teacher to notice and point out to students the value of a physics approach (i.e., know and use the discourse of physics).

How teachers and learners approach (a) navigating different discourses, (b) establishing classroom mathematical discourse(s), and (c) the tools they have to do this, are all informed by their intercultural orientation. In pursuit of applications of this model and data analyses, we had several questions for RUME participants in the session.

Question 1 to attendees: What would make a compelling argument for you about the connections among these ideas? What kinds of data do you suggest we compare?

Attendee response 1: Session participants clearly wanted some rich examples in which the ideas were evidenced so that the evidence could be pointed to (and distinguished from evidence of other aspects of MKT). This call for examples led to the addition (the Area 3 result) of the annotated example conversation between Selma and Jamie.

Question 2 to attendees: Based on your experience, what would you expect about connections among the ideas in the model?

Attendee response 2: Attendees generally agreed that a substantive answer to this question would first require the examples called for in response to the first question.

Question 3 to attendees: How would knowing the answer to the questions we ask help teacher preparation, induction, and development? How would they inform collegiate practice of teaching with the adults who are in-service and pre-service teachers?

Attendee response 3: To get at a transition from theory to practice, participants in the session noted that knowing the answers, and having in hand some examples along with the model and ideas behind Figure 2, gives teacher educators tools and language for instruction (of both pre- and in-service teachers). Also, having an example that gets at the calculus/physics context could allow a contrasting cases approach to understanding the model for teachers. One might create a learning activity for teachers where they start with the calculus/physics discourse analysis (since the difference in the two professional discourses of math and physics may be more accessible to the highly mathematically trained). Then, have a second case where the nuances of analysis are applied to an examination of an example where there are similar professional cultures but differing intercultural orientations. The addition to this report of the Selma and Jamie case arose from the conference conversation. We have also begun development of a contrasting case about two teachers working on, and building a rubric for grading, an applied mathematics item with rich contrasts between physics and mathematical discourse.

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