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Fostering College Students' Autonomy in Written Mathematical Justification

Shandy Hauk

University of Northern Colorado

and

Matthew A. Isom

Arizona State University

Corresponding Author: Shandy Hauk

Mailing Address: Dept. of Mathematical Sciences, Campus Box 122

University of Northern Colorado, Greeley, CO 80639

Phone: 970.351.2344 Fax: 970.351.1225 Email: hauk@unco.edu

Abstract

This study investigated the influence of regular structured writing about problem solving on college algebra students' locus of control, flexibility of articulation, and accuracy in responding to written problem tasks. The writing assignments, acronym PSOLVE, provided students a framework for expressing their thoughts about mathematical actions, processes, structures, and language. Given that augmenting traditional college algebra classes with the PSOLVE writing assignments resulted in statistically significantly higher scores on routine problems on the common post-test ($p < 0.05$), we examined PSOLVE and non-PSOLVE students' written mathematical justifications in response to mildly non-routine short-answer items on the same post-test. PSOLVE students' post-test explanations were clear and mathematically consistent. The PSOLVE writing assignment appears to be a useful support for growth of declarative and procedural knowledge as well as an effective conduit for the instructor to gain insight into students' thinking. We discuss potential benefits of the PSOLVE augmentation for the development of research and practice in college mathematics teaching.

Fostering College Students' Autonomy in
Written Mathematical Justification

Language is acknowledged as important in coming to understand mathematical concepts (Esty, 1992; Morgan, 1998; National Council of Teachers of Mathematics, 2000). Writing across the curriculum and writing-to-learn studies have indicated that the more that material is manipulated through writing, the more it is likely to be remembered and articulated (Gopen & Smith, 1990; Langer, 1992; Pugalee, 2004). When students write about mathematical concepts they attend to both the content and to their understanding of it (Morgan, 1998; Williamson & McAndrew, 1987). Engagement in reflective problem-based questioning has been indicated as a way to enhance mathematics learning in several settings (King, 1994; Mevarech & Kramarski, 2003; Schoenfeld, 1992). In particular, Mayer (1980) demonstrated that in the case of simple, one sentence, word problems in algebra, the view of problem solving commonly held by K-16 instructors – first translate, then solve – was not effective for student learning. Rather, working on algebraic word problems requires a synthesis of translation and reflection abilities. Some have argued that the writing process itself may activate cognitive and meta-cognitive engagement with mathematics that supports problem solving, particularly for strategizing in a novel problem situation (Baker & Czarnocha, 2002).

The stability of lecture-based teaching assumptions in college mathematics service courses such as college algebra, liberal arts mathematics, and mathematics for prospective elementary school teachers makes wholesale implementation of writing-rich problem-based or activity-based learning at the college level a challenge. Nonetheless, the view in K-12 education that successful teaching is evidenced by student learning of procedural *and* conceptual knowledge has made inroads in college settings. The national average of a 30% to 50% failure rate in college algebra courses is some indication that a shift to more constructivist approaches in college mathematics service course teaching may be appropriate – or at least, may be unlikely to make things any worse (Small, 2002). One method for encouraging construction of meaningful understanding about mathematics, even in an otherwise lecture-based course, may be through students writing about their mathematical problem-solving and receiving feedback from their instructors on that writing (Gopen & Smith, 1990; Hiebert & Carpenter, 1992; Morgan, 1998). Both transactional writing (writing to communicate) and expressive writing (reflections on perceptions) are important facets of developing *articulation* in mathematics (Borasi & Rose, 1989; Clarke, Waywood, & Stephens, 1993; Fennema & Romberg, 1999). However, do reflective examination or transactional writing increase students' *performance* in meaningful use of declarative and procedural skills? How might such writing about mathematics problem-solving play a part in students' *conceptual understanding*?

Many researchers have addressed these questions, in a variety of elementary, secondary, and tertiary settings, using myriad theories of learning (Drake & Amspaugh, 1994; Langer, 1992; Morgan, 1998; Porter & Masingila, 2000; Pugalee, 2004; Shepherd, 1993). At the college level, one apparent anomaly is the study of college calculus students by Porter and Masingila (2000). They attempted to characterize the differences in mathematical understanding between students in a class using writing prompts for explaining concepts and those in a class using discussion prompts for the same topics. The authors reported no significant differences in procedural or conceptual understanding between the two groups. Most of the students in Porter and

Masingila's study had taken calculus in high school and were already familiar with a wide range of algebraic concepts, basic calculus terms, and skills. By comparison, a similar study among high school algebra students (Pugalee, 2004), indicated a statistically significant difference between groups. The students who used writing (rather than only discussion) did better on both routine and more complex problems. The students in Pugalee's study were not experienced with the language, symbol sets, and representational standards of algebra. That is, among the notable differences between the two studies was the level of familiarity with mathematical representation and communication by the students.

Like Porter and Masingila's calculus students, college algebra students have taken the course in high school (e.g., second-year high school algebra). However, college algebra students tend not to be pursuing science, mathematics, or engineering degrees and are unlikely to have the same collection of mathematically-rich science course experiences as calculus students. In fact, because many college students in mathematics service courses may have long struggled with mathematics – experiencing it as a mystifying collection of disconnected rules to be memorized (Ellsworth & Buss, 2000; Hauk, 2005) – they may be more like Pugalee's participants.

Though there are policy statements about the value of writing in college mathematics services courses, there is little research on the nature of the influence of out-of-class writing about problem-solving among college algebra learners. Can college algebra students, and their teachers, benefit from students writing about their mathematical efforts? In what way(s)? In particular, the question investigated by this study was: What is the nature of the benefit (if any) to students and teacher if traditional lecture-based instruction in college algebra is augmented by structured regular, outside of class, writing assignments about problem-solving?

As has been the case in other studies, we included a pre-test/post-test statistical comparison of performance on routine problems (i.e., items involving direct application of procedures and skills) and non-routine problems (i.e., items requiring strategizing, planning and execution of multiple steps, or explaining a new method for solution). The focus in this report is on the nature of the conceptions communicated by students on a particular non-routine problem and the ways in which this communication may be beneficial.

Prior to this study, the second author created and piloted a rubric of prompts for the written analysis of problem situations by college learners. For this study, we implemented it in college algebra. In preparing the writing rubric and designing the research, key theoretical influences included constructivist assumptions about the importance of autonomous thinking and communication (von Glasersfeld, 1989), the nature of constructed understanding as a process of interiorization, condensation, and reification (Sfard, 1992), and the dynamic transitions among types of mathematical understanding (Pirie & Kieren, 1994).

Theoretical Perspective and Framework

The epistemological foundation for the study had two grain sizes. On the scale of cognitive theory, the work was informed by the constructivist model of Sfard (1992). Sfard's three stages – interiorization, condensation, and reification – in the learning of mathematics concepts may be used to understand the dualistic nature of procedural and meta-cognitive efforts in transactional writing about mathematical problem-solving. Interiorization of repeated calculational actions into meaningful procedures may be facilitated by asking students to generate examples and express definitions in their own words (Hiebert & Carpenter, 1992).

Condensation and interconnections among the processes involved in a concept (e.g., factorization and the zero product rule in college algebra) may be supported by students' meta-cognitive efforts to explain how they arrived at their answer and to examine and report on the cognitive links between steps in their problem-solving process. Reifying, or organizing related mathematical ideas into a cognitively cohesive structure (a schema), also may be facilitated by writing, especially if the writing prompt calls for such structuring explicitly. For example, interconnected structuring of declarative, procedural, and conceptual knowledge may be promoted by asking the writer to reflect on and discuss why the problem might be assigned and to examine how a particular problem is related to others by creating another, similar, problem whose solution would be arrived at by using the same general method. In particular, conceptual and meaningful procedural understanding may be developed through assignments that call for problem-posing about the concepts in question (Pirie, 2002). However, writing assignments employed without teacher engagement and feedback have not proven any more useful to either students or teachers than traditional mathematics assignments (Gopen & Smith, 1990; Hirsch & King, 1983). Also, as von Glasersfeld (1989) noted, for communication between mathematical expert (teacher) and novice (student) it is important that a shared lexicon for translation exist, otherwise "teaching is likely to remain a hit-or-miss affair."

The time frame under consideration was relatively short (one semester), so the work was also informed by a finer-grained theory. On the scale of examining individual student efforts at constructing understanding, we relied on Pirie and Kieren's (1994) eight-layered model, along with the dynamic of "folding back" among layers. The evidence for comparing the growth of reflection and autonomy among students relied heavily on qualitative analysis of students' written work on pre- and post-tests using the nested components to Pirie and Kieren's recursive theory (see Figure 1).

The inner-most four layers are unlikely to be clearly articulated by a learner, though the shadows of activity at these levels, especially the outer two, may be evidenced in the content and focus of student utterances. In *Primitive Knowing* the word primitive "does not imply low level mathematics, but is rather the starting place for the growth of any particular mathematical understanding. It is what the observer, the teacher or researcher assumes the person doing the understanding can do initially" (Pirie & Kieren, 1994). *Image Making* is the state of creating an internalized image: a representation that can be used in place of something that once may have been in the learner's perceptual field. The result is a state of *Image Having*, where the constructed image is accessible and comparisons to it can be made. *Property Noticing* is doing just that: noting properties of a collection of facts, a procedure, concept, image, or problem situation. Unlike the less clearly articulated inner four levels, the outer four levels typically can be talked about by college learners (though with struggle at times). In *Formalizing* the learner is first able to clearly discuss in detail the properties noticed at the previous level. *Observing* involves comparisons and contrasts - analysis - of formalized understandings. With *Structuring* comes synthesis of observations. The outermost layer, *Inventizing*, includes the capacity to create new ideas (invent) and connect together old ideas in new ways that are based on previous understandings, but are not constrained by them.

Progress through the framework for a student tackling a problem task may be highly non-linear. A student attempting to formalize understanding may struggle with articulation to such an extent that he or she folds back to an inner level of activity (e.g., Property Noticing or Image

Having) as a sense-making strategy. Consideration of the task at a more basic level may result in more densely connected understanding that can then be examined and talked about from an outer level (e.g., Formalizing) perspective by the student. By this means, new understanding can migrate from inner-level thought to the level of Formalizing or Structuring. Sfard's (1992) interiorization, condensation, and reification can be seen as the cognitive processes at work when a student successfully folds back and forth across the boundaries, respectively, between Image Making and Image Having, Property Noticing and Formalizing, and Observing and Structuring.

Many of the students whose work was examined in this study were operating within the innermost five layers of the model throughout their college algebra experience. Their teachers, on the other hand, were so familiar with college algebra that the content of the course had become Primitive Knowing for the construction of other mathematical understandings.

Method

Writing Assignment Rubric

The writing assignment was structured by a problem-solving writing rubric developed and piloted in an earlier study by the second author. The elements of the structured problem solving assignments were:

- **P**roblem restatement: Using complete sentences, state the problem and its main objective.
- **S**olution: Solve the problem, check and explain/justify the solution (if called for). [traditionally, the entirety of student written response for college algebra problems].
- **O**bjective identification and discussion: State the objective of the problem. Why do you think such a problem is assigned? What does one learn from this problem?
- **L**inks: Link the mathematical objects used in deriving your solution from start to finish. That is, construct a generic road map one would follow in finding a solution for a similar problem.
- **V**ocabulary: State two or three vocabulary terms that are directly relevant to this problem or its solution. Provide a brief definition for each term.
- **E**xtend and explain: Create a problem similar to the one you have just solved and provide its solution; explain how the original problem and your problem are similar.

The rubric, acronym PSOLVE, was designed to foster reflection during problem solving and to help students develop a flexible understanding of the mathematics being studied. In the first step, P, stating the main objective in a problem situation promotes reflection by students on what it is they are being asked to do. In step S, solving and sometimes checking are procedures most students are accustomed to performing.

As Borasi and Rose (1989) noted, writing in mathematics has influence on the student as writer, on the teacher as reader, and on the interaction between student and teacher. While college mathematics instructors are usually fluent in the language of mathematics, many college students have difficulty with the densely iconic nature of college mathematics texts. Beyond the traditional solve-and-check, the primary purpose of step O is to open a line of communication between teacher and student based on explicit shared understandings of mathematical objects. This shared understanding, grounded in student word-use and developed over time, is also built through steps L, V, and E. Step L supports students' ability to construct, identify, and communicate their own algorithm(s). Such construction and communication may be linked to the ability to generalize a problem situation to a more abstract structure and is a cornerstone in building sound mathematical habits of mind (Asiala, et al., 1991; Schoenfeld, 1992).

Between the specificity of L and generalization of E lies step V. This part of the PSOLVE activity helps form a shared lexicon for understanding between teacher and student – providing the teacher with student-based words and metaphors that the instructor can encourage students to map onto standard mathematical usage. The process of writing out their understanding of mathematical symbols and other "jargon" provides students an opportunity to internalize the mathematics at hand by organizing the relationships between concepts, words, and symbols. Especially important in these steps may be the opportunity it gives students to contextualize mathematics, an important aspect to constructing understanding (Perret-Claremont & Bell, 1987; White, 2003). The design of the PSOLVE activity rubric was also based on the idea that practice with problem *posing* may play as significant a role in the building of conceptual understanding as problem *solving* (Brown, 2001; Pirie, 2002; Zazkis, Liljedahl, & Gadowsky, 2003). Problem posing is included, in addition to explanation, in step E.

Student Participants

To examine the nature of articulated thought in mathematical problem contexts for students in college algebra, a curricular revision involving PSOLVE assignments was made in eight (of 45) sections of a traditional college algebra course at a large U.S. state research university. The students involved in the study were demographically representative of the institution: 71% European American, 12% Latino American, 5% Asian American, 3% African American, 2% Native American, 3% foreign national (4% unknown)[1]. Most were from public high schools within the same state.

Assignments

Every few weeks in the eight PSOLVE-augmented sections of the course, students were asked to write about their work on a new mildly non-routine problem taken from the textbook (Larson, Hostetler, & Edwards, 1997). These problems all came from the latter part of the exercise sets and were "mildly non-routine" because they required more than a direct application of some procedure from that section of the text. They called for strategy use or planning and coordinating of multiple procedures. A syllabus, including time-line for covering material and a list of suggested homework problems to assign was provided to all instructors by the course coordinator, Ms. Torus. The assigned PSOLVE items came from that list (see Appendix).

Instructor Participants

The implementation of the PSOLVE homework assignments was uneven across the eight randomly selected[2] augmented sections of the course. We chose to focus on the work of students in the classes of the two instructors considered by their teaching peers, and by students, to be the strongest teachers for college algebra in the department. Ms. Torus[3] taught the non-PSOLVE class. She had seven years teaching experience (four at the university, three as a high school teacher), a high school mathematics teaching credential, M.S. degree in mathematics, and was the coordinator for all 45 sections of the course. Mr. Isom (the second author[3]) taught the PSOLVE section. At the time of the study he also had seven years teaching experience (five as a graduate student, two at the university) and a Ph.D. in mathematics education. Both instructors relied mostly on lecture. Before the semester of this study, the pass-rates for students in these instructors' courses were comparable. The performance of their students on previous semesters' common final examinations was also comparable (i.e., there were no statistically significant differences). This was true regardless of the time classes met.

The classes of interest in this study met for an hour each Monday, Wednesday, and Friday in the Fall, 1997, semester. Class meeting time for Mr. Isom's section was around noon, for Ms. Torus it was around 9 a.m. In both cases the classes met in large rooms with florescent lighting, linoleum floors, unadorned walls (except for one wall of chalk boards in each room), and arm-desk chairs (i.e., chairs with small desks, approximately 16" wide by 24" long, fixed on an arm to one side; the arm-desk chairs were not attached to the floor and could be moved).

Pre- and Post-test Instruments and Scoring

Students in all 45 sections of the course completed a common pre-test in the first week of the term and a common post-test during the last week of the semester. The pre-test had 15 free-response items and a five-part matching item (where one column contained questions and a second column contained possible solutions to be matched). The post-test had 8 free-response items, 4 with multiple parts[4]. The authors created a scoring rubric for the pre- and post-test instruments independent of the respective instructors' original grading of the tests. The rubric assigned a zero if the student left a blank space on the test. A score of 1 was assigned if the work offered by the student was incomplete, largely erroneous, or appeared to be irrelevant. A score of 2 was assigned for substantially correct work which either failed to be correct because of an error in calculation, an error through self-reference (e.g., a student might do some work and then make a mistake in reading their own handwriting, transforming a sloppy six in a previous step to a zero in the next step), an error of omission, or stopping before a complete solution was reached. A score of 3 was assigned if the student's work was correct, the solution was correct, and no incorrect or additional work was present (e.g., a student who solved a quadratic involving elapsed time and discovered there were two roots, one of which was negative, would receive a 3 only if the negative solution were clearly rejected; otherwise the score would be 2).

Each of the instruments (pre and post) was graded according to this rubric by the first author. This grading was reviewed by the second author and by a third person, a Ph.D. colleague in the mathematics department with college algebra teaching experience at the university and with experience in mathematics education research. Separately, a research assistant used the rubric to grade all pre- and post-tests. With a few exceptions (on which the grade was arrived at by three-person consensus), the scoring matched, indicating high inter-rater reliability on the scoring of the instruments. The fact that student subscores on comparable items within each test were also comparable indicates internal reliability. According to the instructors who designed the tests (including Torus and Isom), they had face validity (i.e. "the test 'looks valid' to the examinees who take it, the administrative personnel who decide on its use, and other technically untrained observers." Anastasi, 1988; p. 144). A comparison of topics tested to the syllabus and problem-sets for the college algebra course made clear that there was representative content validity. In a test using open-ended questions, internal reliability is difficult to determine with great security. The consistency of scores on similar problems within each instrument was strong. Each idea was tested in some way by at least two items. For example, the pre-test item discussed in detail below was one of three similar problems. A student's scores on these three problems rarely varied by more than one point. Similar, though not as nicely centralized, low variability was found on the post-test (i.e., the variance among like-item scores was low).

The quantitative and qualitative analyses were based on only the pre- and post-test pairs available. The pre-test was administered in all 45 sections of the course during the first week of class. In Mr. Isom's class there were 38 students who took the pre-tests and 35 who took the post-test. However, only 30 of these were students who had begun the term with him (eight of the original students had dropped and five who added the course did so after the first week; they added college algebra and were not transferring from another section of the course). In Ms. Torus' class there were 38 who took the pre-test and 32 who did the post-test. Thirty of these were students who had begun the term with her (two added the class after the first week). Of this 30, three were absent the day of the pre-test so that there were 27 students in her class for whom we had paired pre- and post-tests. Student work on pre- and post-tests was analyzed statistically with a standard statistical analysis package and qualitatively by way of constant comparative methods that coded and categorized student work (Miles & Huberman, 1994).

Integrating the PSOLVE Assignments into the Course

The PSOLVE curricular cycle consisted of three stages for the instructor. First, areas of content focus that appeared to be key for students were noted and an assignment created. Mr. Isom chose the central PSOLVE assignments, from the list of homework problems created by Ms. Torus, and provided the list to all eight instructors who were using PSOLVE. The assignments concerned the topics of slope, composition of functions, quadratic functions and modeling, and exponential modeling (see Appendix A for exact assignments from the text, Larson, Hostetler, & Edwards, 1997). In the second stage of the PSOLVE curricular cycle, students wrote up and turned in responses to an assignment. These write-ups provided instructors with a glimpse into students' understanding and insight into (or struggles with) a problem situation. Finally, in the third stage, the teacher responded to student written work and students had the opportunity to revise their write-ups.

Each student individually completed her or his own collection of PSOLVE assignments. These writing exercises were assigned and collected at the same times as weekly homework exercise sets from the text. Student re-writing, in response to teacher comments on the first draft of each PSOLVE submission, was reviewed by the instructor within two weeks of the original due date. Students' final grades with respect to all of their PSOLVE writings, their "portfolio," were based on the collection of rewritten work and made up 10% of their course grade.[5] Each student submitted a complete portfolio at the last class meeting. Appendix A gives the minimal PSOLVE collection (of five problems) used by the eight instructors augmenting their courses as well as the three additional PSOLVE assignments Mr. Isom chose to use in response to the needs he perceived among his students. As can be seen in Appendix A, the PSOLVE work was presented to students as a type of assignment that was distinct from "regular homework."

Homework assignments for the PSOLVE and traditional courses were comparable in the following sense: if the non-PSOLVE sections had an assignment of 12 problems then the PSOLVE sections' homework assignment was 7 problems with the addition of one PSOLVE problem. Though the PSOLVE problems were from the textbook, the nature of PSOLVE assignments meant students spent more time working with at least five (for Mr. Isom's class, with eight) of these problems than students in the non-PSOLVE classes.

In what follows, work by students in Mr. Isom's PSOLVE augmented course is compared to that by students in Ms. Torus' course in two ways. We begin with a brief discussion of the results of quantitative analysis of student performance on common, free response, pre-test and post-test. However, it is qualitative analysis of the form, length, rigor, and internal consistency of written justifications in problem solutions on pre- and post-tests that constitutes the main body of the results.

Results

The students in PSOLVE sections started the semester with a lower average score on the common pre-test than the non-writing group (not statistically significant). Nonetheless, there was a statistically significant difference in achievement on the common post-test: the PSOLVE students scored higher ($p < 0.05$). The same was also true in a direct comparison of student scores in Mr. Isom's and Ms. Torus' classes ($p < 0.05$). Given this quantitative result, the question of interest became: In what way(s) might students' written explanations and justification be influenced by having used PSOLVE?

The research included examination of all items on both pre- and post-tests. To illustrate what emerged regarding the differences between the groups we report, in particular, on analysis of student written work on problems that focused on logical sense-making about algebraic statements. After presenting information about the Primitive Knowing evidenced by the pre-test items, we give a detailed analysis of the nature of the strategies offered by students for solving a particular moderately non-routine problem on the common post-test. This analysis is framed in the language of the Pirie and Kieren model.

Three categories related to students' devising and using problem-solving strategies emerged from analysis of the pre- and post-test items: locus of control, autonomy, and flexibility of articulation. The problem-solving work offered by students in the PSOLVE and traditionally taught (no writing) groups, combined with the higher performance on basic algebraic skills by the PSOLVE students on the common post-test, made it apparent that at the end of the semester the procedural and conceptual understandings of the traditional course students were not as robust as those of the PSOLVE group. In particular, the pre- and post-test responses of Ms. Torus' students indicated little change over the course of the semester. For students in Mr. Isom's class, several changes could be seen from pre- to post-test: an increase in efforts at articulation, more assertion of autonomous thinking, and broader appeals (to more than teacher, text, and tradition) in justification. There was a great deal of overlap in the strategies used by the students in the two classes. However, there were also strategies distinct to each of the groups.

Where Students Began: Pre-test

On the pre-test students in both classes tended to justify their answers by appeals to authority (teacher, text, and "the rules") or by lengthy, nonsensical or dead-end, algebraic manipulations. On the post-test, Ms. Torus' class exhibited the same tendency. Students in Mr. Isom's PSOLVE class who did make appeals to teacher, text, or tradition on the post-test also tended to fold their own sense-making efforts into their responses; this was accompanied by a notable absence of algebraic rambling.

Three pre-test problems used for comparison of student work in the category "logical sense-making about algebraic statements" are shown below. These items were presented as a group in the pre-test:

Are the following equalities true or false? Explain.[6]

1. $(a+b)^2 = a^2 + b^2$

2. $\sqrt{a^2+b^2} = a+b$

3. $\frac{x^2-4}{x+2} = \frac{x-4}{2}$

The responses of all students to these pre-test items were exemplified by awkward, incomplete, or wandering attempts at explanation. As an example, Table 1 summarizes the breadth of student response for Item 1: approximately one-third of the students in each class incorrectly asserted that the equality was true. The other two-thirds of each class correctly identified the equality as false. However, explanatory efforts were either absent (4 students in each class), incomplete (8 students in each class), or marginally comprehensible (7 in Isom's class, 4 in Torus').

The two classes did not differ significantly in their apparent mastery or their difficulties in communicating an understanding of the basic algebraic pre-requisites for the course. That is, the students in the two classes appeared, qualitatively, to have approximately the same Primitive Knowing. This observation is further reinforced, quantitatively, by the absence of any statistically significant difference in student pre-test scores between the two sections (though Mr. Isom's section did have the lower of the two averages).

Where Students Ended: Post-test

By comparison, student responses to an associated post-test question varied between Ms. Torus and Mr. Isom's classes. The post-test item whose responses we report here in detail:

True or false? To receive credit for this problem you must explain your reasoning.
If $(7x\text{-}3)(2x\text{-}5)=3$, then $(7x\text{-}3)=3$ or $(2x\text{-}5)=3$.

On this post-test item, approximately one-third of each class (Torus: 8/27; Isom: 10/30) appeared to be operating at a Formalizing level in their understanding of the concept tested: asserting the statement was false and justifying their answers by articulating noticed properties of solutions. Examples of correct solutions from each class are discussed in the next three subsections. Most of the students who correctly answered the question solved the quadratic $(7x+3)(2x\text{-}5)=3$ for x and substituted at least one of the values obtained for x into the linear equations to show the solution did not satisfy either linear equality. However, the details of this process were qualitatively different for the two groups.

Solution strategies – Locus of control

Several distinct solution strategies were apparent in student attempts to address the "explain your reasoning" prompt. First we describe these strategies and the ways in which student appeals to authority (their own or that of the text or teacher) were revealed in their strategies. These appeals to authority are aspects of locus of control for reasoning (Schunk, 1999). The description of strategies is followed by a discussion of the flexibility of articulation and accuracy of strategy use by students. Below, the *category name* is followed by the corresponding proportion of students in each class identified as being in that category. Each student response was assigned to exactly one category (see also Figure 10).

Meandering vocabulary – Isom, 2/30 (7%); Torus, 7/27(26%). Students using this strategy provided a plethora of mathematical terms in their attempts to respond to "explain your reasoning." In most cases, the terms were not clearly linked or appropriate to the question. The result was a sort of mathematical jargon dump (see Figure 2). Most of these students correctly asserted that the statement was False. We identified these responses as evidence of some Image Having about real number properties and that students were noticing something about algebraic properties but had not connected or condensed these processes.

Meandering algebra – Isom, 4/30 (13%); Torus, 4/27(15%). This strategy relied on plenty of calculations, frequently in the form of incomplete mathematical statements and often without clear purpose or clear understanding. This was a sort of calculation dump on the part of a student (see Figure 3). Half of the students in this category said True, the other half did not give a True or False answer. We identified such responses as being evidence of students who had not interiorized their Primitive Knowing or Image Making around a meaning for the symbols used.

"You can't do that," The Zero Product Rule – Isom, 4/30 (13%); Torus, 7/27(26%). A presumptive appeal to authority appeared in many student responses. Some were empty of reference to a particular property and others called on the zero-product-rule (ZPR).[7] Figure 4 shows a typical response in this category. Students appeared to be involved in Property Noticing to the extent that the problem's surface features, including its mathematical syntax, was similar to what they were accustomed to dealing with through the use of the ZPR. In other words, students noticed that if the statement had read "If $(7x-3)(2x-5)=0$, then $(7x-3)=0$ or $(2x-5)=0$," then the statement would have been true, and one justification might have been to cite the ZPR. Students using the "You can't do that" strategy appeared to see no difference between necessary and sufficient conditions for invoking the ZPR, demonstrating difficulties with the conventional syntax of mathematics. As seen in the study by Dubinsky & Yiparaki (2000), students relied on their natural language habits, including noticing semantic similarities between a given problem situation and one that might be solved with a known/memorized method (like the ZPR) and attempted to reconcile the two. Unfortunately, though the ZPR holds if the product of linear factors is zero, it is not necessary for the product to equal zero for a similar statement to be true. It is entirely possible for the product of two linear factors to be 3 where one linear factor equals 3 and the other does not.[8] In fact, it is the existence of the conditional algebraic relationship within the "If..., then..." statement that makes this problem moderately non-routine.

"Sure, you can do that," The Three Product Rule – Isom, 4/30 (13%); Torus, 0/27(0%). In place of the "You can't do that" strategy was the creation by some PSOLVE students of a new rule, the "three-product-rule" (TPR): if the product of two factors is 3 then at least one of the factors is equal to 3. It is worth noting that each of the students who created this rule had also used the ZPR, correctly, earlier on the post-test in responding to a routine problem that asked students to identify the zeroes of a quadratic equation with integer roots. The TPR is exemplified in the student response shown in Figure 5. Using the TPR strategy, students created a necessary condition for the statement to be True (rather than relying on the insufficient ZPR to justify an assertion that the statement was False). Clear in such responses was a non-standard use of "zeroes of an equation." Many students used the phrase as a synonym for "solution of an equation," even when the equation in question had not been set equal to zero. Although students asserting the new three-product-rule exhibited an autonomy of approach, there was no evidence

of reflection on whether or not such an argument as justification was connected to standard mathematical rules. As above with the ZPR strategy, the TPR strategy involved Property Noticing and echoed Dubinsky and Yiparaki's (2000) report of students struggling to "experiment [in a mathematical context] in order to assess a mathematical statement." However, with the TPR-centered "Sure, you can do that" strategy, unlike the ZPR-based "You can't do that" approach, student explanations were usually consistent with what was given in the problem and did justify their assertion that the problem statement was True. The "You can't do that" strategy, on the other hand, contained the inconsistency of an assertion that the statement about a product being three was False and justification through an appeal to a rule (the ZPR) that required the product to be zero.

Solution substitution – Isom, 2/30 (7%); Torus, 1/27(4%). Some students solved one of the equations for *x* and then substituted solution(s) from it into one or more other expressions. The follow-through on this strategy was fraught with computational and logical errors. In particular, most errors arose around the mathematical meaning of the word "or" in the problem. Student difficulties in parsing the "or" led to conclusions based on *incomplete* Formalizing: a student might correctly work out solutions to the two linear equations but conclude, after testing only one of the linear equation solutions (by substituting it into the quadratic), that because it did not "work" in the quadratic, the problem's assertion must be false. That is, they appeared to believe conjunctively that both linear solutions would have to solve the quadratic for the problem statement to be true. An example of this kind of strategy is shown in Figure 6.

"3 x 3 = 9" – Isom, 5/30 (17%); Torus, 3/27(11%). Students using this strategy noted that if both linear factors were equal to three then the product of those factors must be nine (or that "it would be larger" than 3 – see Figure 7). As was the case in the solution substitution approach, it appeared students did not or could not attend to the word "or" in the problem statement. In each case, the student clearly treated "or" as "and." Such conflation of meaning for disjunction and conjunction by students, at all ages, has been noted several times in the literature (Damarin, 1977; Hoyles & Küchemann, 2002; Vest, 1981). Also collected into this category was the solution strategy that asserted that the only way a product could equal three was if the factors were strictly less than three. In both modes of this strategy, students tried to reason about the factors as number-like objects rather than as variable algebraic expressions. This might be seen as evidence of an incomplete condensation, where students folded back to Image Having about a process – factoring – in two different contexts and were Property Noticing about factoring without sufficient meta-cognition about any transfer between contexts.

"Three is prime" – Isom, 3/30 (10%); Torus, 1/27(4%). Among the conceptual patterns that arose in student work was the notion that if the product of two linear algebraic factors was three (as in the hypothesis of the problem) then one of the factors must equal 3 and the other must equal 1. That is, students applied prime decomposition concepts (not a topic in the course) for the factoring of whole numbers to the factoring of polynomials. In fact, when carried out carefully, this strategy did result in a correct solution through a loose sort of proof by contradiction (see Figure 8). That is, students were moving toward reification as they were Formalizing. Three things were especially interesting about the "three is prime" solution strategy. First, though students using this strategy did appeal to rules, they did so by calling upon a rule from outside the content of the course. That is, students using this strategy were synthesizing knowledge in a procedurally meaningful way. Secondly, the arguments used by

students were logically consistent (unlike the ZPR justification). Thirdly, students who effectively used the "three is prime" strategy were generating a proof. A complete and correct use of the strategy required the kind of Observing and Structuring involved in a proof by contradiction.

Solution comparison – Isom, 5/30 (17%); Torus, 4/27(15%). Much student work in this category involved Formalizing calculated solutions to at least two equations and some included Observing and Structuring through discussion of the nature of various potential solutions. The (potential) solutions were then compared to determine whether the values for x calculated were consistent with the problem statement's being True or False. As with solution substitution, it was not always clear that students using this strategy understood the significance of the "or" in the item. Consequently, depending on the kinds of algebraic errors that might arise, the strategy was sometimes fairly successful and sometimes not (see Figure 9).

Use of the Strategies – Flexibility of articulation

That the group of students who completed PSOLVE assignments evidenced more skill at clear articulation was to be expected. After all, they had practiced writing about mathematics. However, the category of "flexibility with articulation" had to do with students' description of their use of strategies, particularly the ones they created for themselves. In Figure 10 are the distributions of the solution strategies for each class. Students in the two classes appeared to have been just about as likely to appeal to the solution comparison strategy. The notable differences in strategy choice are at the base of the bars of Figure 10: meandering strategies, that we identified as hovering at the interface between Image Having and Property Noticing, accounted for 20% of the PSOLVE student responses and for 41% of the non-PSOLVE responses.

Though only two of Mr. Isom's 30 PSOLVE students gave solutions exhibiting a meandering writing style, a quarter of those from Ms. Torus' non-PSOLVE class offered solutions in this category. The other interesting aspect of Figure 10 has to do with locus of control. Not a single student in Ms. Torus' non-PSOLVE class created their own rule in an attempt to answer the "explain your reasoning" prompt while one-eighth of Mr. Isom's PSOLVE students did. Perhaps, the PSOLVE students ran into a difficulty with Image Having. They may have noticed properties that were familiar but not the same as the ZPR, perhaps folding back to make a new image with new properties, then Formalizing it as the TPR. One-quarter of the non-PSOLVE and one-eighth of the PSOLVE students noticed ZPR-like properties and simply invoked the unquestioned authority of the zero product rule.

Use of the Strategies – Accuracy

Of equal importance to flexibility was the frequency of substantially and completely correct solutions. Accurate (i.e., substantially or completely correct) solutions contained no logical inconsistencies or algebraic errors or irrelevancies. In the PSOLVE class six solutions were substantially (4 students) or completely (2 students) correct. In the non-PSOLVE class, three of 27 solutions were substantially (2 students) or completely (1 student) correct. This means, of course, that in the PSOLVE-augmented course only 6/30 (20%) of the students offered accurate solutions to this particular mildly non-routine problem, along with 3/27 (11%) of the students in the traditional course.

Discussion

The qualitative difference between the responses of students from the two courses is that the non-standard mathematical conceptions held by some students were clear in the PSOLVE students' work. For example, a fundamental problem understanding the use of "zeroes" and "or" in mathematical contexts was easily discerned in the articulated, though often erroneous, PSOLVE students' solutions. This was especially clear in student writing using the "Sure, you can do that," "Three is prime," "3 x 3 = 9," and solution comparison strategies. These strategies made up 57% of the PSOLVE class' responses. So, more than half of the PSOLVE class was communicating in enough written detail by the post-test that student interpretation of "zeroes" and "or" could be identified. On the other hand, 67% ("You can't do that," meandering algebra, and meandering vocabulary strategies) of the non-PSOLVE students offered such brief or obscure responses that no clear indication of such issues was apparent.

Primitive Knowing, Image Making, and Image Having

The two meandering strategies seem to indicate understanding at work below the Property Noticing level. For this particular problem situation, 20% of the PSOLVE group and 41% of the non-PSOLVE group offered meandering solutions at Image Having or more basic levels of understanding. These students may have been noticing properties, but what they were and how they were connected might only be conjectured.

Property Noticing and Formalizing

Responses using the solution substitution approach indicated students had noticed some properties but not others (e.g., properties of "or" and "zeros" as mathematical indicators) and were likely to have difficulty with formalizing them. About 7% of PSOLVE and 4% of non-PSOLVE students seemed to be entering the Property Noticing level. These students may have had some needed Primitive Knowing and associated images (e.g., about solving for a variable) but not other understanding (about "or" and "zeros") to which they could fold back and from which they could build completely correct solutions.

Students relying on the "You can't do that" and "Sure, you can do that" methods might be said to be operating largely at Property Noticing level – their connecting of the problem statement with either the zero-product-rule or the new three-product-rule could be taken as an indicator that some properties were noticed though formalization of these noticed properties was still nascent. In each class, then, 26% of the students may have been operating at the Property Noticing level in this problem situation.

As mentioned above in our discussion of the various solution strategies, some approaches exemplified articulate, formalized, levels of understanding. Noticed properties were presented and discussed formally by those students using the solution comparison, "Three is prime," and "3 x 3 = 9" strategies. These indicators of operation at the Formalizing level were present in 44% of the PSOLVE students and in 30% of the non-PSOLVE group.

Beyond Formalizing

Of the students who had come through the PSOLVE-enhanced course, three (10%) offered justifications that seemed to be more holistic, perhaps at the Observing or Structuring level of understanding in Pirie and Kieren's (1994) framework. These included the correct "Three is prime" strategy use shown in Figure 8. No student in Ms. Torus' class gave a solution that could be construed as demonstrating understanding at the Observing level or higher.

Conclusion

The students in the PSOLVE classes were statistically significantly more skilled on routine problems than their non-PSOLVE counterparts. Moreover, as evidenced by the detailed examination of one class group, their solutions were better articulated. The non-standard mathematical conceptions held by students were clear in PSOLVE students' written mathematical justifications. The benefits of PSOLVE were two-fold: students were better at doing routine algebra problems and they communicated their thinking on non-routine problems.

Implications for Learning Theory and Research

We conjecture that PSOLVE augmentation of a traditional college algebra class leads to more robust problem situation images. A problem situation image is like a concept image (Tall & Vinner, 1981), but is based on a problem context rather than a concept (Selden, Selden, Hauk, & Mason, 2000). Among the aspects of a problem situation image are noticed properties, condensation of processes and articulation of their formalized connections, and observations a student might make about them on the way to reification. Being able to generate a particular kind of observation, about what a possible solution *method* might be, may be very valuable in learning to solve problems. In addition to the observation of such tentative solution starts, the ability to pose a simpler, related problem is one of the major problem-solving strategies employed by mathematicians. Whether or not a PSOLVE student's problem situation images include more tentative solution starts is not something our data collection was designed to discover. Perhaps, as argued by Selden, et al. (2000), including a prompt that asks for possible solution starts (before asking students to solve), would activate more mental connections among problem situation images through noticed and formalized understandings and would foster condensation of strategic approaches. Future work could be based on P-T-SOLVE, where T is "Tentative solution starts: before solving it, give at least two different ways you might begin to work on this problem." Such work could also be finer grained, examining the evolution of students' written mathematical justification across the semester on the PSOLVE assignments as well as across quizzes and in-term examinations. Of particular interest would be the nature of student efforts to problematize situations (Hiebert et al., 1996). That is, does the use of PSOLVE foster the ability to see situations as problems to be solved rather than as opportunities to apply mastered procedures? A hint that this might be the case arose in this study when several PSOLVE students resorted to creating the TPR rather than applying the ZPR in responding to a non-routine item, even though they had correctly applied the ZPR on a routine item elsewhere in the post-test.

Another obvious place for further research is to follow students from a PSOLVE augmented course into another collegiate mathematics class. The question of interest would be: What is the nature of the strategies used by students and what is the form of their articulation? That is, to what extent do the effects of locus of control, flexibility of articulation, and accuracy – reported for the college algebra students in the current study – persist? Also of interest would be the robustness of structure and flexibility of use for any content knowledge that students might continue to use, perhaps as Primitive Knowing, in further mathematics learning.

Implications for Research on College Teaching

Many college algebra instructors are inexperienced teachers. For example, graduate teaching assistants (GTAs) are novice instructors whose advanced mathematical thinking requires the use of college algebra as Primitive Knowing. It may be that it is quite difficult to treat an entire subject as primitive, in this sense, from 9 to 10 a.m. in a graduate complex analysis course and then suddenly switch perspectives – to college algebra content as something that must be subjected to Property Noticing, Formalizing, Observing, and Structuring for the purpose of teaching it – from 10 to 11 a.m. For this study we concentrated on instructors who had completed their mathematical graduate degrees and had several years experience teaching college algebra. It would be interesting to know the consequences, in terms of both student achievement and college teaching ability development, for GTAs implementing PSOLVE in the undergraduate courses they teach.

In particular, a study of novice college instructors implementing PSOLVE (or PTSOLVE) might also use the Pirie and Kieren (1994) framework to describe the development of college instructors' pedagogical content knowledge (PCK). PCK is the complex structure of content knowledge, syntactic knowledge (of forms of mathematical representation and communication), anticipatory knowledge (of possible challenges and/or smooth transfer points for someone building their understanding of mathematical content), and knowledge for enacting teaching strategies in the classroom (see, for example, Ball & Bass, 2000).

Implications for College Teaching Practice

Through regular PSOLVE assignments, instructors may have access to clear messages about what students do not know because the students learn to write clearly about what they think they know. Despite national policy guidelines and efforts in K-12 schools to implement mathematics process standards in communication and reasoning (National Council of Teachers of Mathematics, 2000), many students still stumble through college algebra. Students seem to learn a kind of visitor's guidebook to mathematical language and standard usage by trial and error, mastering complete phrases that work in particular situations (e.g., using the ZPR as an explanation and justification in a situation where it is neither; or creating a unique conjugation of it, like the three-product-rule reported here). The use of PSOLVE homework assignments in this study had the added value of increasing fluency and articulation of students in their written responses to quiz and exam questions. As mentioned above, this increased ability among undergraduate students to say how they were thinking may facilitate a growth in pedagogical content knowledge for the instructor. For this reason, we feel that PSOLVE may be a very useful tool in helping new college faculty, particularly graduate teaching assistants (GTAs), to develop their teaching skills. Rather than noticing that "Gee, all these students are just wrong" or "everyone gets it because they all say 'false' (in spite of nonsensical explanations offered)" the novice instructor presented with PSOLVE student writing has an opportunity to discern where student difficulties lie. In fact, for any instructor, knowing more about a student's non-standard interpretation of mathematical concepts allows for the development of effective classroom and curricular response.

A novice college teacher may view college algebra as primitive and needing no examination. Implementation of PSOLVE may challenge such an instructional view by exposure to the sincere and articulate struggles of undergraduates to structure their understanding. This, of course, assumes an educational perspective and intention on the part of the instructor to create and maintain a constructivist learning environment[9] (Wilson, 1996). The second author's experiences indicate that folding PSOLVE assignments into college algebra facilitates a constructivist learning environment. The work from PSOLVE students can fulfill expectations for clarity in communication (both generally and about mathematics in particular). In his work with GTAs, the second author also has seen new college teachers become quite interested in PSOLVE as a pedagogical tool, something they can use right away and that requires more thorough cognitive responses from their students. Especially appealing about PSOLVE is that it is a simple addition to lecture-based teaching that requires some feedback from the instructor but that can be implemented without a major investment of class meeting time.

Even if students work in groups to complete PSOLVE problems, the very nature of the assignment provides a framework for discourse and verbalization of justifications. One value in reflective problem solving practice is that it encourages students to take a mental step back from themselves and reflect on their successful, and unsuccessful, efforts. What is more, because computers have the capacity to keep records of process, a computer-assisted version of PSOLVE (or PTSOLVE,) could keep track of the good, bad, and ugly in students' PSOLVE efforts (individually or as teams). Such an audit trail of student work could, in itself, become a useful meta-cognitive tool for students, a way of making explicit the usually tacit paths they trace out in attempting to solve, explain, and extend problems. Computer feedback, in addition to instructor feedback, on problem-solving might enhance interiorization, condensation, and reification of procedural and conceptual understandings as they develop.

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Appendix A. The PSOLVE assignment.

This handout is to be used as a model to follow when analyzing selected mathematics problems. The PSOLVE acronym stands for Problem, Solution, Objects, Links, Vocabulary, and Extensions. After each PSOLVE assignment has been completed, turned in, graded, and returned you will be asked to rework the assignment fixing any errors. Each of the individual PSOLVE exercises will be assigned and collected at the same time the regular homework problems are assigned and collected from the relevant sections of the text. The PSOLVE assignments need to be turned in separately. Your final grade (10% of your course grade) with respect to your PSOLVE portfolio will be based on the collection of all rewritten work. The completed portfolio of all the writing assignments will be collected during the last class meeting in April. NO LATE WORK WILL BE ACCEPTED.

P: State the main objective of this problem in a complete sentence.

S: Solve the problem, check and explain the solution (if required).

O: From either the original statement of the problem or in the mathematics you used in deriving your solution identify any mathematical objects that are new to your understanding or are objects which you do not understand (e.g., formulas, functions, concepts, geometric properties, symbols, etc.)

L: Link the mathematical objects used in deriving your solution from start to finish. That is, construct a generic road map one would follow in finding a solution for a similar problem. Example: point and slope → point-slope formula → slope intercept form → y-axis intercept.

V: State two or three vocabulary terms that are directly relevant to this problem or its solution. In your own words provide a brief definition for each vocabulary term you have identified.

E: Extend and explain: create a similar problem and provide its solution, explain how the original problem and your problem are similar.

PSOLVE	TOPIC
Section 1.2 #50	Slope
Section 1.4 #57	Maximizing Area
Section 1.6 #65	Composition of functions
Section 3.1 #57	Quadratic Height vs. Distance
Section 4.5 #36	Exponential Population Growth

Additions made by Mr. Isom

Section 2.1 #72	Similar Triangles
Section 3.4 #14	Linear Factors
Section 5.1 #59	Cost and Revenue

Note: Section and problem numbers are from Larson, Hostetler, & Edwards (1997).

Table 1. Summary of Student Response to Pre-test Item 1

True	False
35% of student responses	65% of student responses
A. No explanation (1/3 of “True” responses). B. “True, because of the distributive property.” C. “Yes, multiplication is communative.” D. “True, because each variable was squared.” E. “True, both properties raised to the 2nd power - basically just got rid of the parantheses.” F. “True, each item inside the parenthesis is taken to the power outside.” G. “True, a^2+b^2=a^2+b^2”	Calculation shown: $(a+b)^2=a^2+2ab+b^2$ followed by: A. No explanation (1/4 of “False” responses). B. “$a^2+2ab+b^2 = a^2+b^2$.” C. “False, it is not squared properly.” D. “False. Example: a=2, b=3, 5^2=25 but 2^2+3^2=13.” E. “No, $(a+b)^2$ is a 'foil' problem.” F. “False, what is in parentheses always goes first.” G. “False, they won’t factor out to be equal.” H. “False, not all alphabets are multiplied.”

Figure 1. Pirie and Kieren's (1994) nested framework for mathematical understanding.

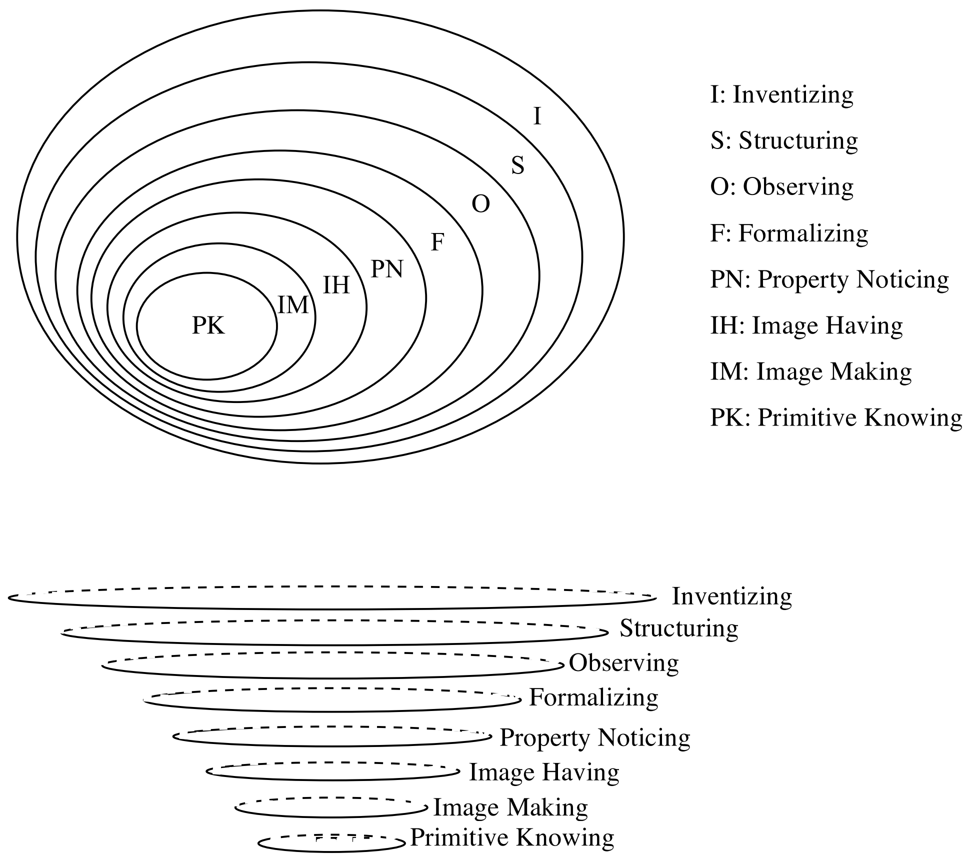


Figure 2. Meandering vocabulary.

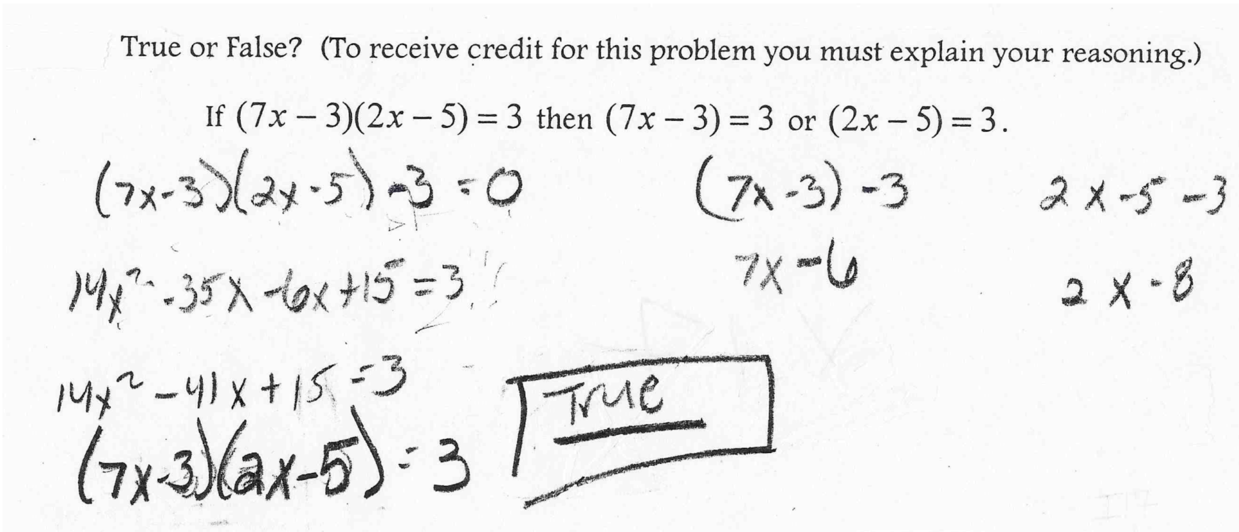
True or False? (To receive credit for this problem you must explain your reasoning.)

If $(7x-3)(2x-5)=3$ then $(7x-3)=3$ or $(2x-5)=3$.

True

Each set of parenthases is a root of the equation (parabola). 2nd degree equation crosses x-axis twice max.

Figure 3. Meandering algebra.



True or False? (To receive credit for this problem you must explain your reasoning.)

If $(7x-3)(2x-5)=3$ then $(7x-3)=3$ or $(2x-5)=3$.

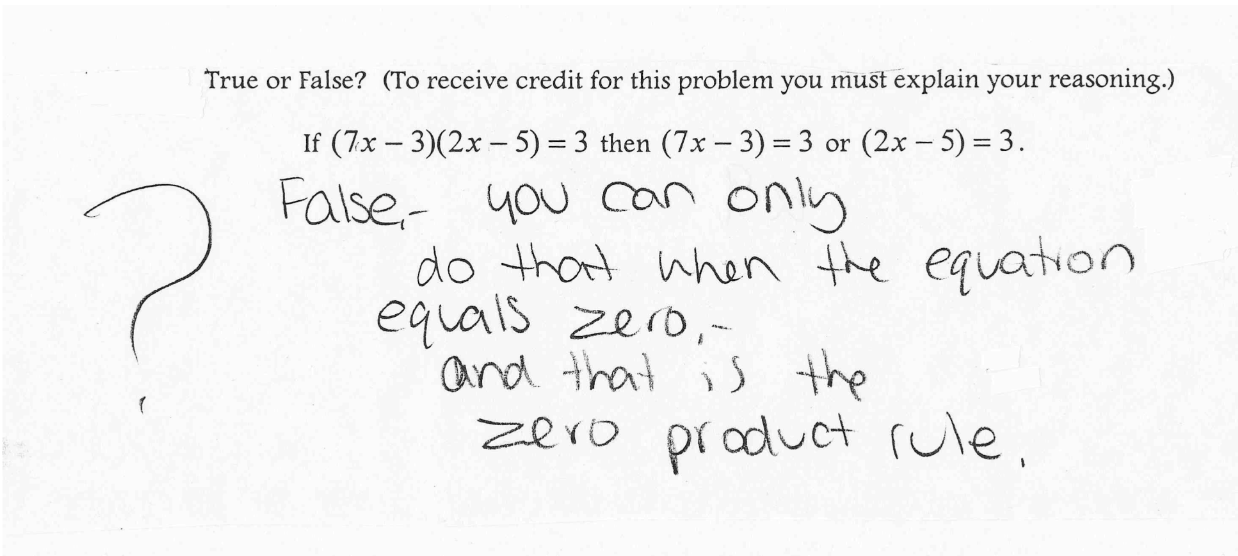
$(7x-3)(2x-5)-3=0$ $\quad (7x-3)-3 \quad 2x-5-3$

$14x^2-35x-6x+15=3$ $\quad 7x-6 \quad 2x-8$

$14x^2-41x+15=3$

$(7x-3)(2x-5)=3$ $\quad \boxed{\text{True}}$

Figure 4. The zero-product-rule.



True or False? (To receive credit for this problem you must explain your reasoning.)

If $(7x-3)(2x-5)=3$ then $(7x-3)=3$ or $(2x-5)=3$.

?

False.- you can only do that when the equation equals zero.- and that is the zero product rule.

Figure 5. The “three-product-rule.”

True or False? (To receive credit for this problem you must explain your reasoning.)

If $(7x-3)(2x-5)=3$ then $(7x-3)=3$ or $(2x-5)=3$.

True, to find the zeros of these linear factors you set each part equal to 3. If the three did not exist each part would be equal to zero etc.

Figure 6. Correct substitution attempt with apparent "and" for "or" error.

True or False? (To receive credit for this problem you must explain your reasoning.)

If $(7x-3)(2x-5)=3$ then $(7x-3)=3$ or $(2x-5)=3$.

$14x^2-35x-6x+15=3$ $\quad \frac{7x}{7}=\frac{6}{7}$ $\quad \frac{2x}{2}=\frac{8}{2}$

$14x^2-41x=-12$ $\quad x=4$

False, the "3" is not the result for both of them.

$14(\frac{6}{7})^2-41(\frac{6}{7})=-12$ ×

$14(4)^2-41(4)=12$
$=60$

$14x^2-41x+12$

$14(\frac{6}{7})^2-41(\frac{6}{7})+12=0$ ×

$14(4)^2-41(4)+12=0$ ×

Figure 7. “3 x 3 = 9” strategy.

True or False? (To receive credit for this problem you must explain your reasoning.)

If $(7x-3)(2x-5)=3$ then $(7x-3)=3$ or $(2x-5)=3$.

False because you have to use the foil method and the answers won't come out to whole numbers. each side will not equal the other one. if $(7x-3)=3$ and $(2x-5)=3$ then the product of these two equations will not also be 3 it would be larger.

Figure 8. "Three is prime" strategy as an informal proof by contraction.

True or False? (To receive credit for this problem you must explain your reasoning.)

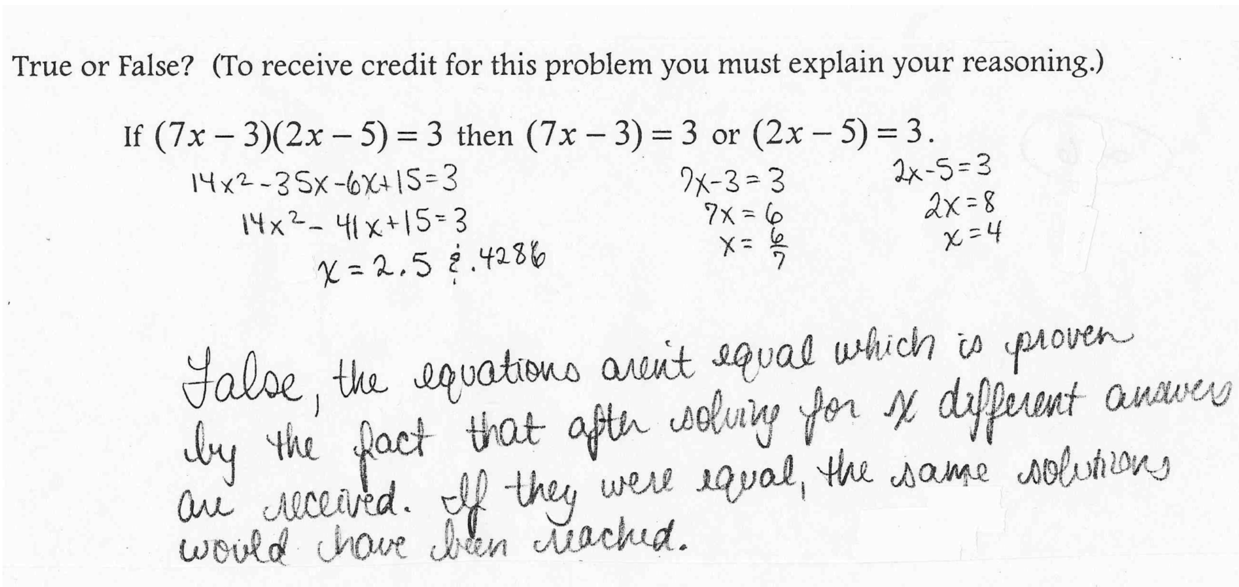
If $(7x-3)(2x-5)=3$ then $(7x-3)=3$ or $(2x-5)=3$.

False, If x makes one set equal to 3, the other will not equal 1.

$(7x-3)=3$
$7x=6$
$x=\frac{6}{7}$
$2(\frac{6}{7})-5=-3.286$
not 1

$(2x-5)=3$
$2x=8$
$x=\frac{8}{2}=4$
$7(4)-3=25$
not 1

Figure 9. Solution comparison method.



True or False? (To receive credit for this problem you must explain your reasoning.)

If $(7x-3)(2x-5)=3$ then $(7x-3)=3$ or $(2x-5)=3$.

$14x^2-35x-6x+15=3$

$14x^2-41x+15=3$

$x=2.5$ & $.4286$

$7x-3=3$

$7x=6$

$x=\frac{6}{7}$

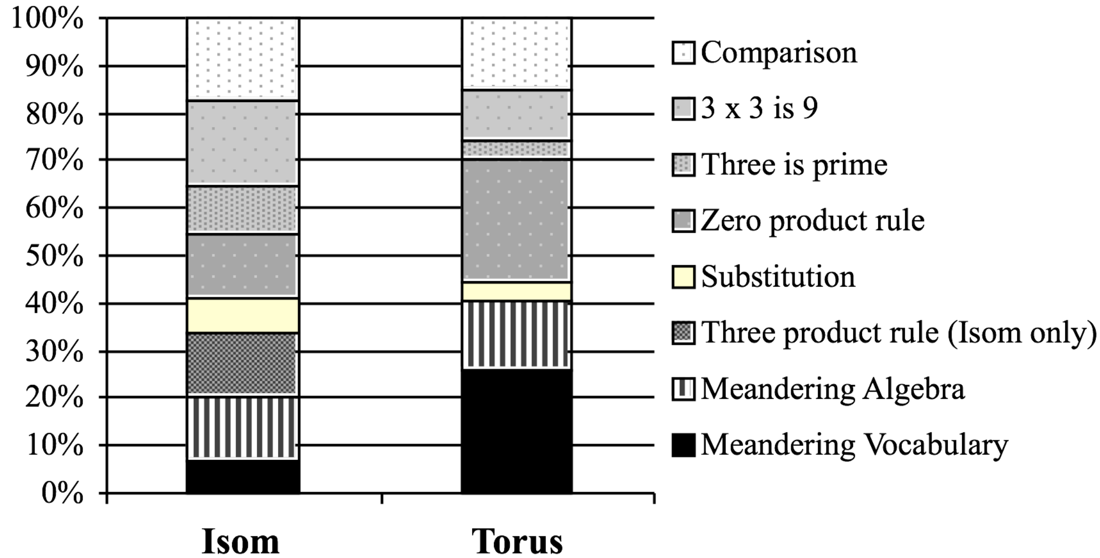
$2x-5=3$

$2x=8$

$x=4$

False, the equations aren't equal which is proven by the fact that after solving for x different answers are received. If they were equal, the same solutions would have been reached.

Figure 10. Distribution of solution methods in the two classes.



Notes

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1 The demographic labels used by the university on entrance forms.

2 Eight course section numbers were drawn from a collection of numbered tickets mixed in a small cardboard box. The second author's was one of these names. All who were approached to participate accepted; however, by the end of the semester only four had regularly used PSOLVE assignments throughout the term.

3 The name “Torus” is a pseudonym.

4 For example, given the coordinates of points *P*, *Q*, and *R*, students were asked to:
(a) Find the equation of the line through *P* and *Q*;
(b) Find the equation of the line through *R* that is perpendicular to the line containing *P* and *Q*;
(c) Algebraically find the point of intersection for the two lines found in parts (a) and (b);
(d) Sketch the answers to (a) and (b) on the same axes and label all intercepts and points of intersection.

5 Homework exercises made up another 10% of the class grade. Together PSOLVE and homework were 20% of the course grade in PSOLVE classes. In non-PSOLVE classes, homework was 20% of the course grade.

6 It is worth noting here that the wording of this, and all other problems, was agreed upon by the group of instructors for college algebra who wrote the instruments. Like most mathematicians and graduate students, these instructors were not trained in test writing. The ambiguity inherent in such questions is another aspect of mathematical communication that students must somehow master.

7 The zero product rule: If the product of factors is zero then at least one of the factors is zero.

8 For example, “If $(7x - 1)(-7x + 5) = 3$, then $(7x - 1) = 3$ or $(-7x + 5) = 3$” is true.

9 A constructivist learning environment is “a place where learners may work together and support each other as they use a variety of tools and information resources in their guided pursuit of learning goals and problem-solving activities” (Wilson, 1996).

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